Kinetic Equations

Solution to the Exercises

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Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T] \times \mathbb{R}^d\right)$ be bounded with $\operatorname{div}_x b$ bounded. Assume that $u_0 \in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ and $f \in L^1\left([0,T]; L^1_{\text{loc}}\left(\mathbb{R}^d\right)\right)$.

Prove that there exists a unique function $u \in L^\infty\left([0,T]; L^1_{\text{loc}}\left(\mathbb{R}^d\right)\right)$ such that for any $\varphi \in C^\infty_{\text{c}}\left([0,T] \times \mathbb{R}^d\right)$ the map $t \mapsto \langle u(t,\cdot), \varphi\rangle$ is continuous in t and which is solution to

$$\begin{cases} \partial_t u + b \cdot \nabla_x u = f, & \text{in } \mathcal{D}'\left([0,T] \times \mathbb{R}^d\right), \\ u|_{t=0} = u_0, & \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \end{cases} \tag{1}$$

Proof. First of all, given that $b \in C^1\left([0,T] \times \mathbb{R}^d\right)$, then there exist $X(s,t,x) \in C^1\left([0,T] \times \mathbb{R}^d\right)$ solution to

$$\begin{cases} \partial_s X(s,t,x) = b(s, X(s,t,x)), & \forall (s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d, \\ X(t,t,x) = x, & \forall (t,x) \in [0,T] \times \mathbb{R}^d \end{cases} \tag{2}$$

Using Duhamel's formula, we define the function v as

$$v(t,x) := u_0(X(0,t,x)) + \int_0^t ds\ f(s, X(s,t,x)). \tag{3}$$

From the hypotheses on f and b we have that v is well defined almost everywhere in x and t; moreover $v \in L^\infty\left([0,T]; L^1_{\text{loc}}\left(\mathbb{R}^d\right)\right)$. Indeed let K be a compact in $\mathbb{R}^d$; we get

$$\sup_{t\in[0,T]} \int_K dx\ v(t,x) \leqslant \sup_{t\in[0,T]} \left[\int_K dx\ \left|u_0(X(0,t,x))\right| + \int_K dx\ \int_0^t ds\ \left|f(s, X(s,t,x))\right|\right] \tag{4}$$

$$= \sup_{t\in[0,T]} \left[\int_{X(t,0,K)} dx\ \left|u_0(x)\right| J(t,0,x)\right. \tag{5}$$

$$\left. + \int_0^t ds\ \int_{X(t,s,K)} dx\ \left|f(s,x)\right| J(t,s,x)\right]. \tag{6}$$

On the one hand we have that for any point $x \in K$, using the fact that b is bounded, we get

$$|X(s,t,x)| = \left|x - \int_s^t dr\ b(r, X(r,t,x))\right| \leqslant |x| + \|b\|_\infty T. \tag{7}$$

As a consequence the set $X(t,s,K) \subseteq K + B_{\|b\|_\infty T}(0)$ is bounded and therefore compact (it is closed by the continuity of X). On the other hand we have that J is uniformly bounded, indeed

$$J(s,t,x) = e^{-\int_s^t dr\ (\mathrm{div}_x b)(r,X(r,t,x))} \leqslant e^{\|\mathrm{div}_x b\|_\infty T}. \tag{8}$$

We therefore get

$$\sup_{t\in[0,T]} \int_K dx\ v(t,x) \leqslant e^{\|\mathrm{div}_x b\|_\infty T} \int_{K+B_{\|b\|_\infty T}(0)} dx\ \left[|u_0(x)| + \int_0^t ds\ |f(s,x)| \right], \tag{9}$$

and $v \in L^\infty\left([0,T], L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right)$.

We now prove that v is the unique solution to (1); to do so, we first show that $t \mapsto \langle v(t,\cdot), \varphi \rangle$ is continuous; first notice that

$$\int_{\mathbb{R}^d} dx\ \varphi(x)\, h(X(s,t,x)) = \int_{\mathbb{R}^d} dx\ \varphi(X(t,s,x))\, h(x)\, J(t,s,x). \tag{10}$$

Now, both X and J are continuous functions, and using the dominated convergence theorem we get that $t \mapsto \langle v(t,\cdot), \varphi \rangle$ is continuous for every $\varphi \in C_c^\infty\left([0,T] \times \mathbb{R}^d\right)$. Fix now $\varphi \in C_c^\infty\left([0,T] \times \mathbb{R}^d\right)$; we get

$$\langle \partial_t v + b \cdot \nabla_x v, \varphi \rangle = \tag{11}$$

$$= -\int_0^T dt \int_{\mathbb{R}^d} dx\ v(t,x,)\left(\partial_t \varphi(t,x) + \mathrm{div}_x(b(t,x)\varphi(t,x))\right) \tag{12}$$

$$= -\int_0^T dt \int_{\mathbb{R}^d} dx\ u_0(X(0,t,x))\left(\partial_t \varphi(t,x) + \mathrm{div}_x(b(t,x)\varphi(t,x))\right) \tag{13}$$

$$- \int_0^T dt \int_{\mathbb{R}^d} dx \int_0^t ds\ f(s,X(s,t,x))\left(\partial_t \varphi(t,x) + \mathrm{div}_x(b(t,x)\varphi(t,x))\right). \tag{14}$$

For the first term notice that

$$\int_{\mathbb{R}^d} dx\ u_0(X(0,t,x))\,\partial_t \varphi(t,x) = \int_{\mathbb{R}^d} dx\ u_0(x)\,(\partial_t \varphi)(t,X(t,0,x))\,J(t,0,x) \tag{15}$$

$$= \int_{\mathbb{R}^d} dx\ u_0(x)\left[\partial_t\left(\varphi(t,X(t,0,x))\,J(t,0,x)\right)\right. \tag{16}$$

$$- b(t,X(t,0,x)) \cdot (\nabla_x \varphi)(t,X(t,0,x))\,J(t,0,x) \tag{17}$$

$$\left. -\varphi(t,X(t,0,x))\,(\mathrm{div}_x b)(t,X(t,0,x))\,J(t,0,x)\right] \tag{18}$$

$$= \partial_t \int_{\mathbb{R}^d} dx\ u_0(x)\,\varphi(t,X(t,0,x))\,J(t,0,x) \tag{19}$$

$$- \int_{\mathbb{R}^d} dx\ u_0(X(0,t,x))\,\mathrm{div}_x(b(t,x)\varphi(t,x)). \tag{20}$$

When we do the integration in time, now, the first integral disappears and therefore the term in (13) vanishes. Proceeding analogously we also get

$$\int_{\mathbb{R}^d} dx\ f\left(s, X\left(s,t,x\right)\right) \partial_t \varphi\left(t,x\right) = \tag{21}$$

$$= \int_{\mathbb{R}^d} dx\ f\left(s,x\right) \partial_t \left(\varphi\left(t, X\left(t,s,x\right)\right) J\left(t,s,x\right)\right) \tag{22}$$

$$- \int_{\mathbb{R}^d} dx\ f\left(s, X\left(s,t,x\right)\right) \mathrm{div}_x \left(b\left(t,x\right) \varphi\left(t,x\right)\right). \tag{23}$$

If we do both the integral in s and t for the first term, we get

$$\int_0^T dt\ \int_0^t ds\ \int_{\mathbb{R}^d} dx\ f\left(s,x\right) \partial_t \left(\varphi\left(t, X\left(t,s,x\right)\right) J\left(t,s,x\right)\right) = \tag{24}$$

$$= \int_{\mathbb{R}^d} dx\ \int_0^T ds\ f\left(s,x\right) \int_s^T dt\ \partial_t \left(\varphi\left(t, X\left(t,s,x\right)\right) J\left(t,s,x\right)\right) \tag{25}$$

$$= - \int_{\mathbb{R}^d} dx\ \int_0^T ds\ f\left(s,x\right) \varphi\left(s, X\left(s,s,x\right)\right) J\left(s,s,x\right) \tag{26}$$

$$= - \int_{\mathbb{R}^d} dx\ \int_0^T ds\ f\left(s,x\right) \varphi\left(s,x\right) = -\langle f, \varphi \rangle, \tag{27}$$

and therefore v is a solution to (1) in $\mathcal{D}'\left(\left[0,T\right] \times \mathbb{R}^d\right)$.

Consider now u_1 and u_2 two different solutions to (1) in $\mathcal{D}'\left(\left[0,T\right] \times \mathbb{R}^d\right)$. Then, $u_1 - u_2$ would solve the problem

$$\begin{cases} \partial_t u + b \cdot \nabla_x u = 0, & \text{in } \mathcal{D}'\left(\left[0,T\right] \times \mathbb{R}^d\right), \\ u|_{t=0} = 0, & \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \end{cases} \tag{28}$$

We now show then that the unique solution to this problem is the solution which is constantly 0. Let u be such a solution and define now

$$w\left(t,x\right) := u\left(t, X\left(t,s,x\right)\right) J\left(t,s,x\right). \tag{29}$$

Consider then $\varphi \in C_c^\infty\left(\left[0,T\right] \times \mathbb{R}^d\right)$; we get that, given that u is a solution

$$\langle \partial_t w, \varphi \rangle = - \int_0^T ds\ \int_{\mathbb{R}^d} dx\ w\left(t,x\right) \partial_t \varphi\left(t,x\right) \tag{30}$$

$$= - \int_0^T ds\ \int_{\mathbb{R}^d} dx\ u\left(t, X\left(t,s,x\right)\right) J\left(t,s,x\right) \partial_t \varphi\left(t,x\right) \tag{31}$$

$$= - \int_0^T ds\ \int_{\mathbb{R}^d} dx\ u\left(t,x\right) \left(\partial_t \varphi\right)\left(t, X\left(0,t,x\right)\right). \tag{32}$$

Given that we have

$$\left(\partial_t \varphi\right)\left(t, X\left(0,t,x\right)\right) = \tag{33}$$

$$= \partial_t \left(\varphi\left(t, X\left(0,t,x\right)\right)\right) + \left(b\left(t,x\right) \cdot \nabla_x X\right)\left(0,t,x\right) \cdot \left(\nabla_x \varphi\right)\left(t, X\left(0,t,x\right)\right) \tag{34}$$

$$= \partial_t \left(\varphi\left(t, X\left(0,t,x\right)\right)\right) + b\left(t,x\right) \cdot \nabla_x \left(\varphi\left(t, X\left(0,t,x\right)\right)\right). \tag{35}$$

Using a smoothing argument similar to the one we saw in class and the equation for u we get that

$$\int_0^T ds \int_{\mathbb{R}^d} dx \; u(t,x)\, \partial_t (\varphi(t, X(0,t,x))) = -\langle \partial_t u, \varphi(\cdot, X(0,\cdot,\cdot))\rangle \tag{36}$$

$$= \langle b \cdot \nabla_x u, \varphi(\cdot, X(0,\cdot,\cdot))\rangle \tag{37}$$

$$= -\int_0^T ds \int_{\mathbb{R}^d} dx \; u(t,x)\, b(t,x) \cdot \nabla_x (\partial_t (\varphi(t, X(0,t,x)))) . \tag{38}$$

As a consequence, we get that $\partial_t w = 0$ in $\mathcal{D}'\left([0,T] \times \mathbb{R}^d\right)$. As a consequence w is constant in time, and therefore we get

$$u(s,x) = u(s, X(s,s,x))\, J(s,s,x) = w(s,x) \tag{39}$$

$$= w(0,x) = u(0, X(0,s,x))\, J(0,s,x) = 0, \tag{40}$$

which concludes our proof.

□

Exercise 2

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T] \times \mathbb{R}^d\right)$ be bounded with $\operatorname{div}_x b$ bounded. Assume that $u_0 \in C^1\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$. Thanks to the first exercise, we now that there exists $u \in L^\infty\left([0,T]; L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)\right)$ such that for any $\varphi \in C_c^\infty\left([0,T] \times \mathbb{R}^d\right)$ the map $t \mapsto \langle u(t,\cdot), \varphi\rangle$ is continuous in t and which is a solution to

$$\begin{cases} \partial_t u + b \cdot \nabla_x u = 0, & \text{in } \mathcal{D}'\left([0,T] \times \mathbb{R}^d\right), \\ u|_{t=0} = u_0, & \text{in } \mathcal{D}'\left(\mathbb{R}^d\right). \end{cases} \tag{41}$$

Prove that the following statements are equivalent:

- $u \in L^\infty\left([0,T] \times \mathbb{R}^d\right)$ is a renormalized solution to (41);
- $u \in C^1\left([0,T] \times \mathbb{R}^d\right)$ is a classical solution to (41).

Proof. If u is a renormalized solution in particular it is a weak solution to (41), and we know from the previous point that the solution is unique. On the other hand the solution given by

$$u(t,x) := u_0(X(0,t,x)) \tag{42}$$

is a classical solution to (41), and a fortiori also a weak one. Therefore the unique solution must be classic.

Conversely, let $u \in C^1\left([0,T] \times \mathbb{R}^d\right)$ be a classic solution; consider $\beta \in C^1(\mathbb{R};\mathbb{R})$. We get

$$\partial t (\beta(u(t,x))) = \beta'(u(t,x))\, \partial_t u(t,x) = -\beta'(u(t,x))\, b(t,x) \cdot \nabla_x u(t,x) \tag{43}$$

$$= -b(t,x) \cdot \nabla_x (\beta(u(t,x))), \tag{44}$$

therefore also $\beta(u)$ is a classical solution and a fortiori also a weak one and u is a renormalized solution.

□

Exercise 3

Let Ω_1 and Ω_2 be two measurable spaces with σ-finite measures μ_1 and μ_2 respectively. Let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a $\mu_1 \times \mu_2$ measurable function and assume that $f \geqslant 0$. Let $p \in [1, +\infty)$. Then

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)\, d\mu_2(y)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leqslant \left(\int_{\Omega_1}\int_{\Omega_2} f(x,y)^p\, d\mu_2(y)\, d\mu_1(x)\right)^{\frac{1}{p}} \tag{45}$$

Proof. Given that μ_1 and μ_2 are σ-finite, using Fubini's Theorem we get that the function

$$x \mapsto \int_{\Omega_2} f(x,y)^p\, d\mu_2(y) \tag{46}$$

is measurable for any $p \in [1, +\infty)$, so both sides of the inequality make sense.

We can assume that the left hand side is non-zero; indeed, if it is the inequality is trivial, given that the right hand side is of course positive. We assume moreover for the moment that the quantity on the left is finite. Define now F as the following measurable function:

$$F(x) := \int_{\Omega_2} f(x,y)\, d\mu_2(y). \tag{47}$$

Then the left hand side can be rewritten as

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)\, d\mu_2(y)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} = \left(\int_{\Omega_1} F(x)^p\, d\mu_1(x)\right)^{\frac{1}{p}} \tag{48}$$

We then get

$$\int_{\Omega_1} F(x)^p\, d\mu_1(x) = \int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)\, d\mu_2(y)\right) F(x)^{p-1}\, d\mu_1(x) \tag{49}$$

$$= \int_{\Omega_2}\left(\int_{\Omega_1} f(x,y)\, F(x)^{p-1}\, d\mu_1(x)\right) d\mu_2(y). \tag{50}$$

Given that $p' = \frac{p}{p-1}$ and using Hölder inequality, we get

$$\int_{\Omega_1} f(x,y)\, F(x)^{p-1}\, d\mu_1(x) \leqslant \left(\int_{\Omega_1} f(x,y)^p\, d\mu_1(x)\right)^{\frac{1}{p}} \left(\int_{\Omega_1} F(x)^p\, d\mu_1(x)\right)^{\frac{p-1}{p}} \tag{51}$$

We now plug in this in the estimate for the left hand side to get

$$\int_{\Omega_1} F(x)^p\, d\mu_1(x) \leqslant \left(\int_{\Omega_1} F(x)^p\, d\mu_1(x)\right)^{\frac{p-1}{p}} \int_{\Omega_2}\left(\int_{\Omega_1} f(x,y)^p\, d\mu_1(x)\right)^{\frac{1}{p}} d\mu_2(y), \tag{52}$$

and carrying the first integral to the left side we get the result.

If now the left hand side is infinite, we consider the sequence of functions $f_M := f\chi_M$, where χ_M is the characteristics of the set $\{(x,y) \in \Omega_1 \times \Omega_2 |\ f(x,y) \leqslant M\}$. Moreover, we fix two sequences of sets $\{\omega_1^n\}_{n\in\mathbb{N}}$, $\{\omega_2^n\}_{n\in\mathbb{N}}$ such that $\mu_j\left(\omega_j^n\right) < +\infty$ and $\bigcup_{n\in\mathbb{N}}\omega_j^n = \Omega_j$ for $j = 1, 2$; given that the left hand side of the inequality to prove is finite for f_M on $\omega_1^n \times \omega_2^n$, we get

$$\left(\int_{\omega_1^n}\left(\int_{\omega_2^n} f_M(x,y)\, d\mu_2(y)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \leqslant \left(\int_{\omega_1^n}\int_{\omega_2^n} f_M(x,y)^p\, d\mu_2(y)\, d\mu_1(x)\right)^{\frac{1}{p}} \tag{53}$$

$$\leqslant \left(\int_{\Omega_1}\int_{\Omega_2} f(x,y)^p\, d\mu_2(y)\, d\mu_1(x)\right)^{\frac{1}{p}}. \tag{54}$$

On the other hand, we get

$$\sup_{n\in\mathbb{N}, M\geqslant 0}\left(\int_{\omega_1^n}\left(\int_{\omega_2^n} f_M(x,y)\, d\mu_2(y)\right)^p d\mu_1(x)\right)^{\frac{1}{p}} = +\infty, \tag{55}$$

and therefore the inequality is still true.

□